

A SEMIUNIFORM ERGODIC THEOREM FOR RANDOM DYNAMICAL SYSTEMS

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ABSTRACT. We study skew product systems over a measurable base and provide a random analogue to the classical (Semi-)Uniform Ergodic Theorem for continuous maps on compact metric spaces. As an application, we study systems forced by a combination of random and deterministic noise and provide criteria, in terms of Lyapunov exponents, for the existence of random attractors with continuous structure in the fibres. For this purpose, we also introduce and discuss some basic concepts of random topological dynamics.

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1. INTRODUCTION

Given an ergodic measure-preserving transformation T on a probability space (M, \mathcal{B}, μ) , Birkhoff's Ergodic Theorem ensures the equality of time and space average for any integrable observable $\Phi : M \rightarrow \mathbb{R}$. More precisely, we have

$$(1.1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \Phi \circ T^i(x) = \mu(\Phi) \quad \text{for } \mu\text{-a.e. } x \in M.$$

Here, we write $\mu(\Phi) = \int_M \Phi d\mu$. When M is a compact metric space, Φ is continuous and T is a continuous and *uniquely ergodic* map, the latter meaning that there exists a unique T -invariant Borel probability measure on M , then this statement can be strengthened by replacing pointwise convergence in (1.1) with uniform convergence on M . In fact, instead of unique ergodicity it suffices to have a unique value for the integrals of Φ with respect to T -invariant probability measures. Let $\mathcal{M}(T)$ denote the set of T -invariant Borel probability measures on M .

Theorem 1.1 (Uniform Ergodic Theorem). *Let M be a compact metric space, $T : M \rightarrow M$ a continuous transformation and $\Phi \in \mathcal{C}(M, \mathbb{R})$. Suppose there exists $\lambda \in \mathbb{R}$ such that $\mu(\Phi) = \lambda$ for all $\mu \in \mathcal{M}(T)$. Then the sequence $S_n(\Phi) = \frac{1}{n} \sum_{i=0}^{n-1} \Phi \circ T^i$ converges uniformly to the constant function λ on M .*

A proof can be found, for example, in [8].

In many applications, it is actually not important to have uniform convergence of the averaged Birkhoff sums. Instead, it suffices to have a uniform bound on the $S_n(\Phi)$ from one side. In particular, this concerns situations where uniform contraction or expansion properties of the system are derived from information on the Lyapunov exponents. In this case, the subadditivity of the considered sequence of observables only allows one-sided estimates anyway. In order to give a precise statement, recall that a sequence of functions $\Phi_n : M \rightarrow \mathbb{R}$ is subadditive with respect to T if

$$(1.2) \quad \Phi_{n+m} \leq \Phi_n \circ T^m + \Phi_m \quad \text{for all } n, m \in \mathbb{N}.$$

Given $\mu \in \mathcal{M}(T)$, (1.2) implies

$$(1.3) \quad \mu(\Phi_{n+m}) \leq \mu(\Phi_n) + \mu(\Phi_m),$$

such that by subadditivity the limit $\lim_{n \rightarrow \infty} \frac{1}{n} \mu(\Phi_n) = \inf_{n \in \mathbb{N}} \frac{1}{n} \mu(\Phi_n)$ exists. Kingman's Subadditive Ergodic Theorem also ensures the μ -a.s. existence of the pointwise limit

$$(1.4) \quad \bar{\Phi}(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \Phi_n(x),$$

provided Φ_1 is integrable, and we have

$$(1.5) \quad \bar{\Phi}(x) = \mu(\bar{\Phi}) = \inf_{n \in \mathbb{N}} \frac{1}{n} \mu(\Phi_n) \quad \mu\text{-a.s.}$$

Again, uniform information on the convergence can be obtained for continuous observables on compact metric spaces.

Theorem 1.2 (Semiuniform Ergodic Theorem, [11]). *Let M be a compact metric space, $T : M \rightarrow M$ a continuous transformation and $(\Phi_n)_{n \in \mathbb{N}}$ a subadditive sequence of continuous functions. Suppose for $\lambda \in \mathbb{R}$ we have $\mu(\bar{\Phi}) < \lambda$ for all $\mu \in \mathcal{M}(T)$. Then there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that*

$$\frac{1}{n} \Phi_n(x) \leq \lambda - \delta \quad \text{for all } n \geq n_0, x \in M.$$

Our aim is to provide an analogue to this result for the application to systems forced either by purely random noise or, more importantly, by a mixture of random and deterministic noise. To that end, we have to introduce some notions and terminology concerning random dynamical systems. A *random map with base* $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, in the sense of Arnold [2], is a skew product of the form

$$(1.6) \quad T : \Omega \times M \rightarrow \Omega \times M, \quad T(\omega, x) = (\theta(\omega), T_\omega(x)),$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\theta : \Omega \rightarrow \Omega$ is a bi-measurable and ergodic measure-preserving bijection.¹ Further it is required that the fibre maps $T_\omega : M \rightarrow M$ are continuous and that for each $x \in M$ the map $\omega \mapsto T(\omega, x)$ is \mathcal{F} -measurable.

As before, we denote the set of all T -invariant probability measures on $\Omega \times M$ by $\mathcal{M}(T)$, and the set of $\mu \in \mathcal{M}(T)$ which project to \mathbb{P} by $\mathcal{M}_{\mathbb{P}}(T)$. We call $\Phi : \Omega \times M \rightarrow \mathbb{R}$ a *random continuous function* if

- (i) the functions $x \mapsto \Phi(\omega, x)$ are continuous for all $\omega \in \Omega$;
- (ii) the functions $\omega \mapsto \Phi(\omega, x)$ are measurable for all $x \in M$.

Similarly we say $K \subseteq \Omega \times M$ is a *random compact (closed) set* if

- (i) $K(\omega) = \{x \in M \mid (\omega, x) \in K\}$ is compact (closed) for all $\omega \in \Omega$;
- (ii) the functions $\omega \mapsto d(x, K(\omega))$ are measurable for all $x \in M$.

$U \subseteq \Omega \times M$ is a *random open set* if U^c is random closed. A random set K is called *forward T -invariant* if $T_\omega(K(\omega)) \subseteq K(\theta(\omega))$ for \mathbb{P} -almost every $\omega \in \Omega$. It is called *T -invariant* if the inclusion is replaced by equality. Note that in contrast to random T -invariance, the notions of random continuous functions and random compact sets do not depend on the measure \mathbb{P} .

For any forward T -invariant random compact set K , we denote the set of $\mu \in \mathcal{M}_{\mathbb{P}}(T)$ which are supported on K by $\mathcal{M}_{\mathbb{P}}^K(T)$. Given a random continuous function $\Phi : \Omega \times M \rightarrow \mathbb{R}$ and a random compact set K , we define

$$(1.7) \quad \Phi^{\max}(\omega) := \max\{\Phi(\omega, x) \mid x \in K(\omega)\}.$$

Finally, we call a random variable $C : \Omega \rightarrow \mathbb{R}$ *adjusted to θ* , if it satisfies $\lim_{|n| \rightarrow \infty} \frac{1}{|n|} C(\theta^n \omega) = 0$ for \mathbb{P} -a.e. ω .

Theorem 1.3 (Random Semiuniform Ergodic Theorem). *Let $T : \Omega \times M \rightarrow \Omega \times M$ be a random map with ergodic base $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. Suppose that $(\Phi_n)_{n \in \mathbb{N}}$ is a subadditive sequence of random continuous functions with $|\Phi_n|^{\max} \in L^1(\mathbb{P})$ for all $n \in \mathbb{N}$, and that K is a forward T -invariant random compact set. Further, assume that $\lambda \in \mathbb{R}$ satisfies $\bar{\Phi}_\mu < \lambda$ for all $\mu \in \mathcal{M}_{\mathbb{P}}^K(T)$.*

Then there exist $\lambda' < \lambda$ and an adjusted random variable $C : \Omega \rightarrow \mathbb{R}$ such that

$$(1.8) \quad \Phi_n(\omega, x) \leq C(\omega) + n\lambda' \quad \text{for all } n \in \mathbb{N}, \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and all } x \in K(\omega).$$

In particular, for $\delta \in (0, \lambda - \lambda')$ and \mathbb{P} -a.e. $\omega \in \Omega$ there exists $n(\omega) \in \mathbb{N}$ such that

$$(1.9) \quad \frac{1}{n} \Phi_n(\omega, x) \leq \lambda - \delta \quad \text{for all } n \geq n(\omega) \text{ and } x \in K(\omega).$$

¹The base $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a *measure-preserving dynamical system*.

Remark 1.4. (a) This theorem generalises [11, Theorem 1.9] and [11, Corollary 1.11] from from the deterministic to the random setting. It should be mentioned that [11] also contains a random version of these results (Theorem 1.19), but only in the sense that the statement is made with respect to a fixed reference measure on the base, while the space Ω is still assumed to be compact and both T and Φ_n are required to be continuous. This theorem is therefore not applicable to general randomly forced systems, and the proof avoids the difficulties coming from the lack of a topological structure on Ω that we have to deal with here.

(b) Note that C is adjusted in the above sense if and only if e^C is tempered [2, Definition 4.1.1]. Hence, by [2, Proposition 4.3.3], given $\epsilon > 0$ there is a random variable D_ϵ such that

$$-\epsilon|n| + D_\epsilon(\omega) \leq D_\epsilon(\theta^n \omega) \leq \epsilon|n| + D_\epsilon(\omega) \text{ for all } n \in \mathbb{Z}$$

and $C \leq D_\epsilon$, so that

$$\Phi_n(\omega, x) \leq D_\epsilon(\omega) + n\lambda' \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and all } x \in K(\omega).$$

An important field of applications for the Semiuniform Ergodic Theorem 1.2 are deterministically forced systems. Here, a system under the influence of an external factor, given by a continuous transformation g on a compact metric space Ξ , is modelled as a skew product

$$(1.10) \quad T : \Xi \times \mathbb{R}^d \rightarrow \Xi \times \mathbb{R}^d, \quad T(\xi, y) = (g(\xi), h_\xi(y)).$$

When the fibre maps h_ξ are differentiable with derivative matrix Dh_ξ , then the Lyapunov exponent of an invariant measure $\mu \in \mathcal{M}(T)$ is given by

$$(1.11) \quad \lambda(\mu, T) = \mu(\bar{\Phi}) = \inf_{n \in \mathbb{N}} \frac{1}{n} \mu(\Phi_n),$$

where the sequence of subadditive functions Φ_n on $\Xi \times \mathbb{R}^d$ is given by $\Phi_n(\xi, y) = \log \|Dh_\xi^n(y)\|$. Here we use the notation $h_\xi^n = h_{g^{n-1}(\xi)} \circ \dots \circ h_\xi$. In [11, Theorem 1.14], Sturman and Stark provide a criterion for the continuity of attractors in terms of Lyapunov exponents. More precisely, they show that a compact T -invariant set which only supports invariant measures with negative Lyapunov exponents has to be a finite union of curves.

In Section 3, we apply Theorem 1.3 to provide different random analogues to this statement. The most basic one is the fact that a random compact set which only admits negative Lyapunov exponents (in the above sense) consists of a finite number of points in each fibre (Corollary 3.4). A more sophisticated result concerns systems which are forced by a combination of random and deterministic noise. We model such systems as double skew products of the form

$$(1.12) \quad T : \Omega \times \Xi \times \mathbb{R}^d \rightarrow \Omega \times \Xi \times \mathbb{R}^d, \quad T(\omega, \xi, y) = (\theta(\omega), g_\omega(\xi), h_{\omega, \xi}(y)),$$

where $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is a measure-preserving dynamical system and Ξ is a compact metric space. The second component Ξ corresponds to the deterministic part of the forcing and is allowed to depend on the random noise as well. Note that fibre maps of T acting on $\Xi \times \mathbb{R}^d$ are given by $T_\omega(\xi, y) = (g_\omega(\xi), h_{\omega, \xi}(y))$.

From the point of view of modelling real-world processes, such an overlay of random and deterministic forcing seems very natural. However, to our knowledge such systems have not been studied previously in the literature. What we provide here is a random analogue to [11, Theorem 1.14], which can be circumscribed as follows. If g is a random minimal homeomorphism² and $K \subseteq \Omega \times \Xi \times \mathbb{R}^d$ is a random compact T -invariant set which only admits negative Lyapunov exponents, then for \mathbb{P} -a.e. $\omega \in \Omega$ the set $K(\omega) = \{(\xi, y) \mid (\omega, \xi, y) \in K\}$ is a finite union of curves. The precise result is stated as Theorem 3.3 below. A slightly weaker statement still holds when g is only random transitive (Remark 3.9). If K is connected in each fibre, then the minimality assumption on g can even be dropped completely (Theorem 3.5). Apart from their intrinsic interest, statements of this kind can be useful tools in the theory of random bifurcations. In particular, a specific version of

²See Section 4 for the definition of random minimality.

Theorem 3.5 has already been used in [1] to describe a generalised torus³ splitting off of a stable central manifold in a non-autonomous Hopf (or Neimark-Sacker) bifurcation. For the particular class of considered models, this allows to give a rigorous proof of Arnold's two-step scenario for the random Hopf bifurcation [2, 10].

Finally, some basic facts concerning random minimality and random transitivity are collected and discussed in Section 4. These notions appear quite naturally in our context, but again we are not aware of any similar concepts studied in the literature so far. However, we believe they may play an important role in the development of a theory of random topological dynamics.

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2. THE RANDOM SEMIUNIFORM ERGODIC THEOREM

The aim of this section is to prove Theorem 1.3. We start by providing some more basic facts on random dynamical systems. Let M be a Polish space and let $T : \Omega \times M \rightarrow \Omega \times M$ be a continuous random map over the ergodic measure-preserving dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. This means that (compare [2])

- (i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\theta : \Omega \rightarrow \Omega$ is a bi-measurable bijection that preserves the probability measure \mathbb{P} ;
- (ii) θ is ergodic, that is, $\mathbb{P}(A) \in \{0, 1\}$ for each $A \in \mathcal{F}$ such that $\theta^{-1}A = A$;
- (iii) For each $\omega \in \Omega$, the map $T_\omega : M \rightarrow M$, $x \mapsto T(\omega, x)$, is continuous, and for each $x \in M$, the map $\omega \mapsto T(\omega, x)$ is \mathcal{F} -measurable.

As M is Polish, (iii) implies that T is $\mathcal{F} \times \mathcal{B}$ -measurable where \mathcal{B} is the Borel- σ -algebra of M [5, Lemma III.14].

Recall that given a random continuous function $\Phi : \Omega \times M \rightarrow \mathbb{R}$ and a random compact set K , we let

$$(2.1) \quad \Phi^{\max}(\omega) := \max\{\Phi(\omega, x) \mid x \in K(\omega)\}.$$

The following lemma is easily derived from results in [5], but we include the proof for the convenience of the reader. Note that the related statements [5, Lemma III.39] and [3, Theorem 8.2.11] require completeness of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Lemma 2.1. *$\Phi^{\max} : \Omega \rightarrow \mathbb{R}$ is measurable, and there is a measurable map $h : \Omega \rightarrow M$ such that $h(\omega) \in K(\omega)$ and $\Phi^{\max}(\omega) = \Phi(\omega, h(\omega))$ for all $\omega \in \Omega$.*

Proof. For the random compact set K there is a sequence $(a_k)_{k \in \mathbb{N}}$ of measurable maps $a_k : \Omega \rightarrow M$ such that $K(\omega) = \text{closure}\{a_k(\omega) \mid k \in \mathbb{N}\}$ for all $\omega \in \Omega$ [5, Theorem III.30], see also [2, Proposition 1.6.3]. Therefore, observing the random continuity of Φ ,

$$\Phi^{\max}(\omega) = \max\{\Phi(\omega, x) \mid x \in K(\omega)\} = \sup\{\Phi(\omega, a_k(\omega)) \mid k \in \mathbb{N}\}$$

is measurable.

For $\ell \in \mathbb{N}$ let $k_\ell(\omega) = \min\{k \in \mathbb{N} \mid \Phi(\omega, a_k(\omega)) > \Phi^{\max}(\omega) - \ell^{-1}\}$. The $k_\ell : \Omega \rightarrow \mathbb{N}$ are measurable. Therefore,

$$H(\omega) = \bigcap_{j \in \mathbb{N}} \bigcup_{\ell \geq j} \overline{\{a_{k_\ell(\omega)}(\omega)\}} \subseteq \{x \in K(\omega) \mid \Phi(\omega, x) = \Phi^{\max}(\omega)\}$$

is a random compact set [5, Proposition III.4], and there is a measurable selection $h : \Omega \rightarrow M$ such that $h(\omega) \in H(\omega) \subseteq K(\omega)$ for all $\omega \in \Omega$ [5, Theorem III.9]. \square

The proof of the next lemma is straightforward.

Lemma 2.2. *If the sequence $(\Phi_n)_{n \in \mathbb{N}}$ of random continuous functions is subadditive and K is a forward invariant random compact set, then the sequence $(\Phi_n^{\max})_{n \in \mathbb{N}}$ is subadditive.*

³A random compact invariant set which consists of a topological circle in each fibre.

From now on, we use the hypothesis of Theorem 1.3 as standing assumptions for the remainder of this section. Subadditivity of $(\Phi_n^{\max})_{n \in \mathbb{N}}$ allows to define

$$\overline{\Phi}^{\max} = \inf_{n \in \mathbb{N}} \frac{1}{n} \mathbb{P}(\Phi_n^{\max}) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{P}(\Phi_n^{\max}).$$

Obviously $\overline{\Phi}_\mu \leq \overline{\Phi}^{\max}$ for each $\mu \in \mathcal{M}_{\mathbb{P}}^K(f)$.

The proofs of the following two results are inspired by the proofs of Lemmas 3 and 4 in [4]. Note that any measure $\mu \in \mathcal{M}_{\mathbb{P}}(T)$ can be disintegrated into a family of probability measures $(\mu_\omega)_{\omega \in \Omega}$ on the fibres, in the sense that $\int_{\Omega \times M} \Phi d\mu = \int_{\Omega} \int_M \Phi(\omega, x) d\mu_\omega(x) d\mathbb{P}(\omega)$ for all measurable functions $\Phi : \Omega \times M \rightarrow \mathbb{R}$ [2, Proposition 1.4.3].

Lemma 2.3. *We have $\overline{\Phi}^{\max} = \sup\{\overline{\Phi}_\mu \mid \mu \in \mathcal{M}_{\mathbb{P}}^K(T)\}$, and the supremum is attained by some $\mu^* \in \mathcal{M}_{\mathbb{P}}^K(T)$.*

Proof. Let h_n be measurable selections such that $\Phi_n^{\max}(\omega) = \Phi_n(\omega, h_n(\omega))$, see Lemma 2.1. Define measures $\mu_n \in \mathcal{M}_{\mathbb{P}}^K(T)$ via their fibre measures

$$\mu_{n,\omega}(\theta) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T_{\theta^{-i}\omega}^i(h_n(\theta^{-i}\omega))},$$

where δ_x denotes the Dirac measure in a point $x \in M$. As $h_n(\omega) \in K(\omega)$ for each $\omega \in \Omega$, all measures μ_n are supported by the forward invariant random compact set K . Hence, by the random Krylov-Bogolyubov Theorem ([6] or [2, Theorem 1.6.13]), there is a subsequence $(\mu_{n_l})_{l \in \mathbb{N}}$ converging (random-)weakly to some $\mu^* \in \mathcal{M}_{\mathbb{P}}^K(f)$.

Now fix $k \in \mathbb{N}$. Then, for some $t \in \{0, \dots, k-1\}$ the sequence $(n_l)_{l \in \mathbb{N}}$ contains a subsequence of the form $(s_j k + t)_{j \in \mathbb{N}}$. Note that for any sequence $(x_j)_{j \in \mathbb{N}}$ of measurable functions $x_j : \Omega \rightarrow M$ with $x_j(\omega) \in K(\omega)$ for all $\omega \in \Omega$, $j \in \mathbb{N}$, and any sequence of integers $(N_j)_{j \in \mathbb{N}}$ with $N_j \nearrow \infty$, the fact that $|\Phi_k|^{\max} \in L^1(\mathbb{P})$ easily implies

$$(2.2) \quad \lim_{j \rightarrow \infty} \frac{1}{N_j} \int_{\Omega} \Phi_k(\omega, x_j(\omega)) d\mathbb{P}(\omega) = 0.$$

Using this observation, we obtain

$$\begin{aligned} \mu^* \left(\frac{1}{k} \Phi_k \right) &= \lim_{j \rightarrow \infty} \frac{1}{k} \int_{\Omega} \Phi_k d\mu_{s_j k + t} \\ &= \lim_{j \rightarrow \infty} \frac{1}{k(s_j k + t)} \sum_{i=0}^{s_j k + t - 1} \int_{\Omega} \Phi_k \circ T^i(\theta^{-i}\omega, h_{s_j k + t}(\theta^{-i}\omega)) d\mathbb{P}(\omega) \\ &= \lim_{j \rightarrow \infty} \frac{1}{k(s_j k + t)} \sum_{i=0}^{s_j k + t - 1} \int_{\Omega} \Phi_k \circ T^i(\omega, h_{s_j k + t}(\omega)) d\mathbb{P}(\omega) \\ &\stackrel{(2.2)}{=} \lim_{j \rightarrow \infty} \frac{1}{k(s_j k + t)} \sum_{i=0}^{k-1} \sum_{l=0}^{s_j - 2} \int_{\Omega} \Phi_k \circ T^{kl+i}(\omega, h_{s_j k + t}(\omega)) d\mathbb{P}(\omega) \\ &\geq \lim_{j \rightarrow \infty} \frac{1}{k(s_j k + t)} \sum_{i=0}^{k-1} \int_{\Omega} \Phi_{(s_j - 1)k} \circ T^i(\omega, h_{s_j k + t}(\omega)) d\mathbb{P}(\omega) \\ &\geq \lim_{j \rightarrow \infty} \frac{1}{k(s_j k + t)} \sum_{i=0}^{k-1} \left(\int_{\Omega} \Phi_{s_j k + t}(\omega, h_{s_j k + t}(\omega)) d\mathbb{P}(\omega) \right. \\ &\quad \left. - \int_{\Omega} \Phi_i(\omega, h_{s_j k + t}(\omega)) + \Phi_{k-i+t} \circ T^{(s_j - 1)k+i}(\omega, h_{s_j k + t}(\omega)) d\mathbb{P}(\omega) \right) \\ &\stackrel{(2.2)}{=} \lim_{j \rightarrow \infty} \frac{1}{s_j k + t} \int_{\Omega} \Phi_{s_j k + t}(\omega, h_{s_j k + t}(\omega)) d\mathbb{P}(\omega) \\ &= \lim_{j \rightarrow \infty} \frac{1}{s_j k + t} \int_{\Omega} \Phi_{s_j k + t}^{\max}(\omega) d\mathbb{P}(\omega) = \overline{\Phi}^{\max}. \end{aligned}$$

Since this holds for all $k \in \mathbb{N}$, we have $\overline{\Phi}_{\mu^*} = \inf_{k \in \mathbb{N}} \mu^* \left(\frac{1}{k} \Phi_k \right) \geq \overline{\Phi}^{\max}$. \square

For the proof of Theorem 1.3, we will further need the following useful criterion for the adjustedness of random variables.

Lemma 2.4. *Suppose $C : \Omega \rightarrow \mathbb{R}$ is measurable and $C \circ \theta - C$ has a \mathbb{P} -integrable minorant. Then C is adjusted to θ .*

Proof. Since $C \circ \theta - C$ has an integrable minorant, [9, Lemma 4.1.13] implies that $C \circ \theta - C \in L^1(m)$ and $\int_{\Omega} C \circ \theta - C \, dm = 0$. Hence, we obtain from the Birkhoff Ergodic Theorem that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} C(\theta^n \omega) &= \lim_{n \rightarrow \infty} \frac{1}{n} (C(\theta^n \omega) - C(\omega)) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} C(\theta^{i+1} \omega) - C(\theta^i \omega) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} (C \circ \theta - C) \circ \theta^i(\omega) = 0 \end{aligned}$$

for m -a.e. $\omega \in \Omega$. Since $C \circ \theta^{-1} - C = -(C \circ \theta - C) \circ \theta^{-1} \in L^1(\mathbb{P})$ as well, the limit for $n \rightarrow -\infty$ can be treated in the same way. \square

Now we can turn to the proof of the first main result.

Proof of Theorem 1.3. Suppose $\lambda > \bar{\Phi}_{\mu}^{\max}$ for all $\mu \in \mathcal{M}_{\mathbb{P}}^K(T)$. Then $\lambda > \bar{\Phi}^{\max}$ by Lemma 2.3, and we can choose $\lambda' \in (\bar{\Phi}^{\max}, \lambda)$. We have $\lim_{n \rightarrow \infty} \frac{1}{n} \Phi_n^{\max}(\omega) = \bar{\Phi}^{\max}$ \mathbb{P} -a.s. by Kingman's Subadditive Ergodic Theorem. If we let $\Phi_0^{\max} = 0$, then the random variable C defined by

$$C(\omega) = \sup_{n \geq 0} (-\lambda' n + \Phi_n^{\max}(\omega))$$

is non-negative and \mathbb{P} -a.s. finite. The fact that C satisfies (1.8) is obvious from its definition. Further, if $C(\omega) = 0$ then $C(\theta \omega) - C(\omega) \geq 0$. Otherwise, since

$$\begin{aligned} -\lambda' n + \Phi_n^{\max}(\omega) &\leq (-\lambda'(n-1) + \Phi_{n-1}^{\max}(\theta \omega)) + (-\lambda' + \Phi_1^{\max}(\omega)) \\ &\leq C(\theta \omega) - \lambda' + \Phi_1^{\max}(\omega), \end{aligned}$$

for all $n \geq 1$, we have that

$$C(\omega) \leq C(\theta \omega) - \lambda' + \Phi_1^{\max}(\omega).$$

Combining both estimates yields

$$C(\theta \omega) - C(\omega) \geq \min\{0, \lambda' - \Phi_1^{\max}(\omega)\}.$$

Hence $C \circ \theta - C$ has an integrable minorant, and thus C is adjusted to θ by Lemma 2.4. \square

For the application to forced systems, we will also need the following variation of Theorem 1.3.

Theorem 2.5. *In the situation of Theorem 1.3, there exist $\lambda' < \lambda$ and $k_0 \in \mathbb{N} \setminus \{0\}$ such that for all $k \geq k_0$ there are an adjusted random variable $\hat{C}_k : \Omega \rightarrow [0, \infty)$ and an ergodic component⁴ Ω_k of θ^k with $\mathbb{P}(\Omega_k) \geq 1/k$ such that*

$$(2.3) \quad \Phi_k(\omega, x) \leq \hat{C}_k(\theta^k \omega) - \hat{C}_k(\omega) + k\lambda' \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega_k \text{ and all } x \in K(\omega).$$

The random variables \hat{C}_k can also be chosen to take values in $(-\infty, 0]$. Furthermore

- a) *if $(\Phi_n)_{n \in \mathbb{N}}$ is additive, then $k_0 = 1$ so that (2.3) holds for $k = 1$ and \mathbb{P} -a.e. $\omega \in \Omega$;*
- b) *if θ is totally ergodic,⁵ then (2.3) holds for \mathbb{P} -a.e. $\omega \in \Omega$.*

Proof. As $\bar{\Phi}^{\max} < \lambda$ by Lemma 2.3, there is $k_0 \in \mathbb{N}$ such that $\mathbb{E}_{\mathbb{P}}[\Phi_k^{\max}] = \int_{\Omega} \Phi_k^{\max} \, d\mathbb{P} < k\lambda'$ for some $\lambda' < \lambda$ and all $k \geq k_0$. Fix any such k . Then

$$(2.4) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \Phi_k^{\max}(\theta^{-jk} \omega) = \mathbb{E}_{\mathbb{P}}[\Phi_k^{\max} \mid \mathcal{I}_k](\omega) \quad \text{for } \mathbb{P}\text{-a.e. } \omega,$$

⁴An ergodic component of θ^k is a θ^k -invariant set Ω_k of positive measure such that $\theta|_{\Omega_k}^k$ is ergodic.

⁵That is, θ^k is ergodic for all $k \in \mathbb{N}$.

where $\mathbb{E}_{\mathbb{P}}[\cdot | \mathcal{I}_k]$ denotes the conditional expectation w.r.t. the σ -algebra $\mathcal{I}_k \subseteq \mathcal{F}$ of all θ^k -invariant sets. Since θ is ergodic, all sets of positive measure in \mathcal{I}_k have measure at least $1/k$. As $\int_{\Omega} \mathbb{E}_{\mathbb{P}}[\Phi_k^{\max} | \mathcal{I}_k] d\mathbb{P} = \mathbb{E}_{\mathbb{P}}[\Phi_k^{\max}] < k\lambda'$, this means that there is an ergodic component of θ^k such that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \Phi_k^{\max}(\theta^{-jk}\omega) < k\lambda'$ for \mathbb{P} -a.e. $\omega \in \Omega_k$. Hence,

$$(2.5) \quad 0 \leq \hat{C}_k(\omega) = \sup_{n \geq 0} \left(-\lambda'nk + \sum_{j=1}^n \Phi_k^{\max}(\theta^{-jk}\omega) \right) < \infty \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega_k.$$

Let $\omega \in \Omega_k$. We prove (2.3):

$$\begin{aligned} \hat{C}_k(\theta^k\omega) &= \sup_{n \geq 0} \left(-\lambda'nk + \sum_{j=0}^{n-1} \Phi_k^{\max}(\theta^{-jk}\omega) \right) \\ &\geq \sup_{n \geq 1} \left(-\lambda'nk + \sum_{j=0}^{n-1} \Phi_k^{\max}(\theta^{-jk}\omega) \right) \\ &= \sup_{n \geq 1} \left(-\lambda'(n-1)k + \sum_{j=1}^{n-1} \Phi_k^{\max}(\theta^{-jk}\omega) \right) - \lambda'k + \Phi_k^{\max}(\omega) \\ &= \hat{C}_k(\omega) - \lambda'k + \Phi_k^{\max}(\omega). \end{aligned}$$

This also implies that $\hat{C}_k \circ \theta^k - \hat{C}_k$ has the integrable minorant $-\lambda'k + \Phi_k^{\max}$ so that, in view of [9, Lemma 4.1.13], $\hat{C}_k \circ \theta^k - \hat{C}_k \in L^1_{\mathbb{P}}$ and $\int \hat{C}_k \circ \theta^k - \hat{C}_k d\mathbb{P} = 0$. Thus, Lemma 2.4 implies that \hat{C}_k is adapted to θ^k . In order to show that it is also adapted to θ , let $\ell \in \{0, \dots, k-1\}$. Then Birkhoff's Ergodic Theorem implies that

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} \frac{1}{nk + \ell} \hat{C}_k(\theta^{nk+\ell}\omega) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{k} (\hat{C}_k \circ \theta^k - \hat{C}_k)(\theta^{jk}(\theta^{\ell}\omega)) \\ &= \frac{1}{k} \mathbb{E}_{\mathbb{P}}[\hat{C}_k \circ \theta^k - \hat{C}_k | \mathcal{I}_k](\theta^{\ell}\omega) \quad \text{for } \mathbb{P}\text{-a.e. } \omega. \end{aligned}$$

As $\int \mathbb{E}_{\mathbb{P}}[\hat{C}_k \circ \theta^k - \hat{C}_k | \mathcal{I}_k] d\mathbb{P} = \int \hat{C}_k \circ \theta^k - \hat{C}_k d\mathbb{P} = 0$, it follows that the limit is actually equal to zero for \mathbb{P} -a.e. ω . Hence \hat{C}_k is adapted to θ . The proof for the limit $n \rightarrow -\infty$ proceeds along the same lines.

Observe that the \hat{C}_k are all non-negative. We finally show how to modify the above construction to obtain non-positive \hat{C}_k . To this end let

$$(2.6) \quad \hat{C}_k(\omega) = -\sup_{n \geq 0} \left(-\lambda'nk + \sum_{j=0}^{n-1} \Phi_k^{\max}(\theta^{jk}\omega) \right).$$

As above one shows that there is an ergodic component Ω_k of θ^k such that $-\infty < \hat{C}_k(\omega) \leq 0$ for \mathbb{P} -a.e. $\omega \in \Omega_k$. We prove (2.3):

$$\begin{aligned} \hat{C}_k(\theta^k\omega) &= \inf_{n \geq 0} \left(\lambda'nk - \sum_{j=1}^n \Phi_k^{\max}(\theta^{jk}\omega) \right) \\ &= \inf_{n \geq 0} \left(\lambda'(n+1)k - \sum_{j=0}^{(n+1)-1} \Phi_k^{\max}(\theta^{jk}\omega) \right) - \lambda'k + \Phi_k^{\max}(\omega) \\ &\geq \hat{C}_k(\omega) - \lambda'k + \Phi_k^{\max}(\omega). \end{aligned}$$

Again, $\hat{C}_k \circ \theta^k - \hat{C}_k$ has the integrable minorant $-\lambda'k + \Phi_k^{\max}$, and as before one proves that \hat{C}_k is adjusted to θ . \square

3. APPLICATIONS TO RANDOM DYNAMICAL SYSTEMS

In order to apply Theorem 1.3 to forced systems, we start by reviewing the result of Sturman and Stark on the continuity of attractors for deterministically forced systems [11, Theorem 1.14]. In their situation, the external forcing is given by a continuous transformation g on a compact metric space Ξ . Writing $M = \Xi \times \mathbb{R}^d$ and denoting elements of M by $x = (\xi, y)$, the forced system then has the form of a skew product

$$(3.1) \quad T : M \rightarrow M \quad , \quad T(\xi, y) = (g(\xi), h_\xi(y)) .$$

When the *fibre maps* h_ξ are \mathcal{C}^r and the partial derivatives $D^\alpha h_\xi(y)$ all depend continuously on (ξ, y) , we call T a *g-forced \mathcal{C}^r -map*. In this case, the sequence

$$(3.2) \quad \Phi_n(\xi, y) = \log \|Dh_\xi^n(y)\|$$

is subadditive with respect to T . Here Dh_ξ denotes the derivative matrix of the map $h_\xi : M \rightarrow M$ and $h_\xi^n = h_{g^{n-1}(\xi)} \circ \dots \circ h_\xi$. The *maximal Lyapunov exponent* of an invariant measure $\mu \in \mathcal{M}(T)$ is defined as

$$(3.3) \quad \lambda(\mu, T) = \mu(\bar{\Phi}) = \inf_{n \in \mathbb{N}} \frac{1}{n} \mu(\Phi_n) .$$

Using the Semiuniform Ergodic Theorem, Sturman and Stark provided the following criterion for the existence of continuous attractors. We denote by $\mathcal{K}(\mathbb{R}^d)$ the space of compact subsets of \mathbb{R}^d , equipped with the Hausdorff distance.

Theorem 3.1 (Sturman/Stark). *Let $g : \Xi \rightarrow \Xi$ be a minimal homeomorphism and T a g -forced \mathcal{C}^1 -map on $M = \Xi \times \mathbb{R}^d$. Further, assume $K \subseteq M$ is a compact T -invariant set and $\lambda(\mu, T) < 0$ for all $\mu \in \mathcal{M}(T)$ supported on K . Then K is a union of continuous curves. More precisely, we have*

- (i) *there exists an integer $n \in \mathbb{N}$ such that $\#K(\xi) = n$ for all $\xi \in \Xi$ and*
- (ii) *the map $\Xi \rightarrow \mathcal{K}(\mathbb{R}^d)$, $\xi \mapsto K(\xi) = \{y \in \mathbb{R}^d \mid (\xi, y) \in K\}$ is continuous.*

By adding additional random noise to (3.1), we obtain random dynamical systems with a double skew product structure

$$(3.4) \quad T : \Omega \times M \rightarrow \Omega \times M \quad , \quad T(\omega, \xi, y) = (\theta(\omega), g_\omega(\xi), h_{\omega, \xi}(y)) ,$$

where $M = \Xi \times \mathbb{R}^d$ as above and Ξ is again a compact metric space. We assume that the maps g_ω are homeomorphisms, the maps $(\xi, y) \mapsto h_{\omega, \xi}(y)$ are continuous and differentiable in y and $Dh_{\omega, \xi}(y)$ is continuous in (ξ, y) for all $\omega \in \Omega$. The action of the g_ω on the second component Ξ corresponds to the deterministic part of the forcing, such that the combined forcing process is the *random homeomorphism*

$$(3.5) \quad \theta \times g : \Omega \times \Xi \rightarrow \Omega \times \Xi \quad , \quad \theta \times g(\omega, \xi) = (\theta(\omega), g_\omega(\xi)) .$$

In this way, given any $\theta \times g$ -invariant probability measure m we can view $T = (\theta \times g) \times h$ as a random map over the base $(\Omega \times \Xi, \mathcal{F} \times \mathcal{X}, m, \theta \times g)$, where \mathcal{X} denotes the Borel σ -algebra on Ξ . An alternative point of view is to write $T = \theta \times (g \times h)$, thus interpreting T as a random map over the base $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. In this case the fibre maps T_ω have a skew product structure themselves, and $T_\omega : M \rightarrow M$ is a continuous transformation of the form $T_\omega(\xi, y) = (g_\omega(\xi), h_{\omega, \xi}(y))$. Given T as in (3.4) and a random T -invariant set K , we let $K(\omega) = \{(\xi, y) \in M \mid (\omega, \xi, y) \in K\}$ and $K(\omega, \xi) = \{y \in \mathbb{R}^d \mid (\omega, \xi, y) \in K\}$. Then $T_\omega(K(\omega)) = K(\theta\omega)$ and $h_{\omega, \xi}(K(\omega, \xi)) = K(\theta \times g(\omega, \xi))$ (see Lemma 3.7 below).

Similar to above, the Lyapunov exponent of a T -invariant measure μ is defined as

$$(3.6) \quad \lambda(\mu, T) = \mu(\bar{\Phi}) = \inf_{n \in \mathbb{N}} \frac{1}{n} \mu(\Phi_n) ,$$

where now the subadditive sequence Φ_n is given by $\Phi_n(\omega, \xi, y) = \log \|D_y h_{\omega, \xi}^n(y)\|$. As mentioned in the introduction, in order to state an analogue to Theorem 3.1 in this setting we need a random version of minimality.

Definition 3.2. *The random homeomorphism $\theta \times g$ is minimal, if each $(\theta \times g)$ -forward invariant random closed set K obeys the following dichotomy:*

either: $K(\omega) = \Xi$ for \mathbb{P} -a.e. ω ,
or : $K(\omega) = \emptyset$ for \mathbb{P} -a.e. ω .

Theorem 3.3. *Let T be a random map of the form (3.4) and K be a T -invariant random compact set. Suppose that for all $k \in \mathbb{N}$ and all $\epsilon > 0$ there is $r > 0$ such that*

$$(3.7) \quad \sup \{ \Phi_k(\omega, \xi, y) \mid \omega \in \Omega, (\xi, y) \in B_r(K(\omega)) \} \\ \leq \sup \{ \Phi_k(\omega, \xi, y) \mid \omega \in \Omega, (\xi, y) \in K(\omega) \} + \epsilon$$

Then, if $\lambda(\mu, T) < 0$ for all measures $\mu \in \mathcal{M}_{\mathbb{P}}^K(T)$, and if $\theta \ltimes g$ is a random minimal homeomorphism on $\Omega \times \Xi$, there are a (non-random) integer $n > 0$ and a random variable $c(\omega) > 0$ such that, for \mathbb{P} -a.e. $\omega \in \Omega$,

- (i) $\#K(\omega, \xi) = n$ for all $\xi \in \Xi$,
- (ii) the map $\xi \mapsto K(\omega, \xi)$ from Ξ to $\mathcal{K}(\mathbb{R}^d)$ is continuous, and
- (iii) for all $\xi \in \Xi$, any two different points $y, y' \in K(\omega, \xi)$ have distance at least $c(\omega)$.

If $T : \Omega \times \mathbb{R}^d \rightarrow \Omega \times \mathbb{R}^d$ is a random map with base $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ then we can add a trivial component $\Xi = \{\xi_0\}$ and let $g_\omega(\xi_0) = \xi_0$ to apply Theorem 3.3. As $\theta \ltimes g$ is certainly random minimal in this case, this immediately yields

Corollary 3.4. *Suppose $T : \Omega \times \mathbb{R}^d \rightarrow \Omega \times \mathbb{R}^d$ is a random \mathcal{C}^1 -map with base $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ and $K \subseteq \Omega \times \mathbb{R}^d$ is a random compact set such that (3.7) is satisfied and $\lambda(\mu, T) < 0$ for all $\mu \in \mathcal{M}_{\mathbb{P}}^K(T)$. Then there exists an integer n such that $\#K(\omega) = n$ for \mathbb{P} -a.e. $\omega \in \Omega$.*

The minimality assumption on $\theta \ltimes g$ in Theorem 3.3 can also be replaced by requiring connectedness of the sets $K(\omega, \xi)$. We say the random compact set $K \subseteq \Omega \times \Xi \times \mathbb{R}^d$ has *connected fibres* if $K(\omega, \xi)$ is connected for \mathbb{P} -a.e. $\omega \in \Omega$ and all $\xi \in \Xi$.

Theorem 3.5. *Let T be a random map of the form (3.4) and K be a T -invariant random compact set with connected fibres. Suppose that for all $k \in \mathbb{N}$ and all $\epsilon > 0$ there is $r > 0$ such that (3.7) holds and that $\lambda(\mu, T) < 0$ for all measures $\mu \in \mathcal{M}_{\mathbb{P}}^K(T)$. Then for \mathbb{P} -a.e. $\omega \in \Omega$ the fibre $K(\omega)$ consists of a single continuous graph, that is, there is a random continuous function $\phi : \Omega \times \Theta \rightarrow \mathbb{R}^d$ such that $K(\omega) = \{(\xi, \phi(\omega, \xi)) \mid \xi \in \Xi\}$ for \mathbb{P} -a.e. $\omega \in \Omega$.*

Both theorems are consequences of the following more technical proposition. We say a random set $A \subseteq \Omega \times \Xi$ is *non-empty*, if $\{\omega \in \Omega \mid A(\omega) \neq \emptyset\}$ has positive measure. Note that if A is $(\theta \ltimes g)$ -invariant and θ is ergodic, then non-emptiness of A implies that $\{\omega \in \Omega \mid A(\omega) = \emptyset\}$ has measure zero. Similarly, equality of random sets will always be understood as an equality of the fibres for \mathbb{P} -a.e. $\omega \in \Omega$.

Proposition 3.6. *Let T be a random map of the form (3.4) and K be a T -invariant random compact set. Suppose that for all $k \in \mathbb{N}$ and all $\epsilon > 0$ there is $r > 0$ such that (3.7) holds. Then, if $\lambda(\mu, T) < 0$ for all measures $\mu \in \mathcal{M}_{\mathbb{P}}^K(T)$, there are a positive integer n , a random variable $c : \Omega \rightarrow (0, \infty)$ and a non-empty open and $(\theta \ltimes g)$ -forward invariant random set A such that, for \mathbb{P} -a.e. $\omega \in \Omega$,*

- (i) $\#K(\omega, \xi) = n$ for all $\xi \in A(\omega)$,
- (ii) $\sup \{ \#K(\omega, \xi) \mid \xi \in \Xi \} < \infty$, and
- (iii) for all $\xi \in \Xi$, any two different points $y, y' \in K(\omega, \xi)$ have distance at least $c(\omega)$.

Proof of Theorem 3.3. As A is a nonempty forward $(\theta \ltimes g)$ -invariant random set, A^c is a backwards $(\theta \ltimes g)$ -invariant random compact set and $A^c(\omega) \neq \Xi$ for \mathbb{P} -a.e. ω . As $\theta \ltimes g$ is minimal, $A^c(\omega) = \emptyset$ for \mathbb{P} -a.e. ω (see Lemma 4.5(ii) below), and hence $A(\omega) = \Xi$ for \mathbb{P} -a.e. ω . Assertions (i) and (iii) of the proposition together with the compactness of $K(\omega)$ imply the continuity of the map $\xi \mapsto K(\omega, \xi)$ from Ξ to $\mathcal{K}(\mathbb{R}^d)$. \square

Proof of Theorem 3.5. By assertion (ii) of the proposition, together with the connectedness of the fibres of K , there is a subset $\Omega_0 \subseteq \Omega$ of full measure such that $K(\omega, \xi) \subset \mathbb{R}^d$ consists of a single point for all $(\omega, \xi) \in \Omega_0 \times \Xi$. As $K(\omega) \subset \Xi \times \mathbb{R}^d$ is compact, it must be the graph of a continuous map $\phi(\omega, \cdot) : \Xi \rightarrow \mathbb{R}^d$. As $\{\phi(\omega, \xi)\} = K(\omega, \xi)$ is the only possible selection of K , ϕ is measurable [2, Proposition 1.6.3]. \square

Proof of Proposition 3.6. By Theorem 2.5 there exist $\lambda' < 0$, $k \in \mathbb{N} \setminus \{0\}$, an ergodic component Ω_k of θ^k , and an adjusted random variable $\widehat{C} : \Omega \rightarrow (-\infty, 0]$ such that

$$\Phi_k(\omega, \xi, y) \leq \widehat{C}(\theta^k \omega) - \widehat{C}(\omega) + k\lambda' \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega_k \text{ and all } (\xi, y) \in K(\omega).$$

Hence, in view of assumption (3.7), there are $r > 0$ and $\eta > 0$ such that

$$(3.8) \quad \log \|D_y h_{\omega, \xi}^k(y)\| \leq \widehat{C}(\theta^k \omega) - \widehat{C}(\omega) - \eta \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega_k \text{ and all } (\xi, y) \in B_r(K(\omega)).$$

Now we consider T as $(\theta \times g) \times h$. We defer the proof of the following two lemmas to the end of this section.

Lemma 3.7. *K is a random compact set over the base $(\Omega \times \Xi, \mathcal{F} \times \mathcal{B})$, and $h_{\omega, \xi}(K(\omega, \xi)) = K((\theta \times g)(\omega, \xi))$ for \mathbb{P} -a.e. all $\omega \in \Omega$ and $\xi \in \Xi$.*

Next, for $\epsilon \in (0, r]$, define $N_\epsilon(\omega, \xi)$ to be the smallest number of open balls $B_{\epsilon e^{\widehat{C}(\omega)}}(y) \subset \mathbb{R}^d$ centred at points $y \in K(\omega, \xi)$ that are needed to cover the compact set $K(\omega, \xi)$.

Lemma 3.8. *a) The N_ϵ are $\mathcal{F} \times \mathcal{B}$ -measurable.*

b) For each $\omega \in \Omega$, the function $N_\epsilon(\omega, \cdot) : \Xi \rightarrow \mathbb{N}$ is upper semicontinuous.

Fix $\epsilon \in (0, r]$, $\omega \in \Omega_k$ and $\xi \in \Xi$, and denote $N = N_\epsilon(\omega, \xi)$. There are $y_1, \dots, y_N \in K(\omega, \xi)$ such that $K(\omega, \xi) \subset \bigcup_{i=1}^N B_{\epsilon e^{\widehat{C}(\omega)}}(y_i)$. As $h_{\omega, \xi}(K(\omega, \xi)) = K((\theta \times g)(\omega, \xi))$, it follows from (3.8) that

$$K((\theta \times g)^k(\omega, \xi)) \subset \bigcup_{i=1}^N h_{\omega, \xi}^k(B_{\epsilon e^{\widehat{C}(\omega)}}(y_i)) \subset \bigcup_{i=1}^N B_{e^{-\eta \epsilon} e^{\widehat{C}(\theta^k \omega)}}(h_{\omega, \xi}^k(y_i)) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega_k$$

with points $h_{\omega, \xi}^k(y_i) \in K((\theta \times g)^k(\omega, \xi))$. Hence

$$(3.9) \quad N_\epsilon((\theta \times g)^k(\omega, \xi)) \leq N_{e^{-\eta \epsilon}}((\theta \times g)^k(\omega, \xi)) \leq N_\epsilon(\omega, \xi).$$

Consider the restricted system $(\theta \times g)^k|_{\Omega_k \times \Theta}$ and denote the normalised probability measure $\mathbb{P}|_{\Omega_k}$ by \mathbb{P}_k . By Lemma 3.8, there is a subset $\Omega'_k \subseteq \Omega_k$ of full measure such that the random sets $U_{\epsilon, \alpha} = \{(\omega, \xi) \in \Omega_k \times \Xi \mid N_\epsilon(\omega, \xi) < \alpha\}$ are open for all $\alpha \in \mathbb{R}$ and $\epsilon = e^{-p\eta}r$ with $p \in \mathbb{N}$. For measurability purposes we restrict to these countably many values of ϵ from now on. Let

$$n_\epsilon(\omega) = \min\{\alpha \in \mathbb{N} \mid U_{\epsilon, \alpha}(\omega) \neq \emptyset\}.$$

The measurability of n_ϵ follows easily from Lemma 4.1d. Due to (3.9) we have $n_\epsilon(\theta^k \omega) \leq n_\epsilon(\omega)$ for \mathbb{P}_k -a.e. $\omega \in \Omega_k$, and thus the ergodicity of (θ^k, \mathbb{P}_k) implies that all n_ϵ are constant \mathbb{P}_k -a.e. We denote these constant integers by n_ϵ again. By the first inequality of (3.9), $n_\epsilon \leq n_{e^{-\eta \epsilon}}$. But the second inequality of (3.9) implies that also $n_{e^{-\eta \epsilon}} \leq n_\epsilon$ for all $\epsilon \in (0, r]$, so that all n_ϵ coincide. Denote their common value by n .

Using (3.9) again, we see that the random open set $U_{r, n}$ is $(\theta \times g)^k$ -invariant and we have $U_{r, n} = U_{\epsilon, n}$ for all ϵ . Similarly, for each integer $m > n$ the set $U_{r, m}$ is a non-empty $(\theta \times g)^k$ -invariant open random set and $U_{r, m} = U_{\epsilon, m}$ for all ϵ . Since the random compact set $K \cap (\Omega_k \times M)$ is covered by these countably many random open sets $(U_{r, m})_{m \in \mathbb{N}}$, we see that the following hold for \mathbb{P}_k -a.e. $\omega \in \Omega_k$:

- (1) $\#K(\omega, \xi) = n$ for all $\xi \in U_{r, n}$;
- (2) $\sup\{\#K(\omega, \xi) \mid \xi \in \Xi\} < \infty$;
- (3) $d(y, y') \geq c(\omega) := re^{\widehat{C}(\omega)}/2 > 0$ for all $\xi \in \Xi$ and any two different points $y, y' \in K(\omega, \xi)$.

Let $A_k = U_{r, n}$. If $k = 1$, then $A = A_k$ satisfies the assertions of the proposition. Otherwise, we let $A = \bigcup_{i=0}^{k-1} (\theta \times g)^i(A_k)$. Then, as $\bigcup_{i=0}^{k-1} \theta^k(\Omega_k) = \Omega$ up to a set of \mathbb{P} -measure zero, as the g_ω^i are homeomorphisms, and as $h_{\omega, \xi}^i(K(\omega, \xi)) = K((\theta \times g)^i(\omega, \xi))$, assertions (1)–(3) carry over to \mathbb{P} -a.e. $\omega \in \Omega$. \square

Remark 3.9. If the second alternative “ $K(\omega) = \emptyset$ for \mathbb{P} -a.e. ω ” in the definition of minimality (Definition 3.2) is replaced by “ $K(\omega)$ is nowhere dense for \mathbb{P} -a.e. ω ”, $\theta \times g$ is called *transitive*. Random transitivity seems to be a more subtle concept than random minimality;

some of its aspects are discussed in Section 4. Here we only note a version of Theorem 3.3 for random transitive homeomorphisms:

If the random homeomorphism in Theorem 3.3 is not minimal but only transitive, then there are a (non-random) integer $n > 0$, a random $\theta \times g$ -invariant open dense set $A(\omega) \subseteq \Xi$ and a random variable $c(\omega) > 0$ such that, for \mathbb{P} -a.e. $\omega \in \Omega$,

- (i) $\#K(\omega, \xi) = n$ for all $\xi \in A(\omega)$,
- (ii) the map $\xi \mapsto K(\omega, \xi)$ from $A(\omega)$ to $K(\mathbb{R}^d)$ is continuous, and
- (iii) for all $\xi \in \Xi$, any two different points $y, y' \in K(\omega, \xi)$ have distance at least $c(\omega)$.

As in the proof of Theorem 3.3, the result is deduced from Proposition 3.6. The only difference is that the sets $A^c(\omega)$ are no longer empty, but nowhere dense, so that the open sets $A(\omega)$ are dense.

Proof of Lemma 3.7. For each $(\omega, \xi) \in \Omega \times \Xi$, the set $K(\omega, \xi) = \{y \in \mathbb{R}^d : (\xi, y) \in K(\omega)\}$ is compact because $K(\omega)$ is. Denote the metric on Ξ by ρ and define, for each $n > 0$, a metric d_n on $\Xi \times \mathbb{R}^d$ by $d_n((\xi, y), (\xi', y')) = \|y' - y\| + n\rho(\xi', \xi)$. Each d_n generates the product topology on $\Xi \times \mathbb{R}^d$. Further, $\omega \mapsto d_n((\xi, y), K(\omega))$ is measurable for all $(\xi, y) \in \Xi \times \mathbb{R}^d$, and $\xi \mapsto d_n((\xi, y), K(\omega))$ is continuous for all $\omega \in \Omega$ and $y \in \mathbb{R}^d$. It follows that $(\omega, \xi) \mapsto d_n((\xi, y), K(\omega))$ is measurable for all $y \in \mathbb{R}^d$, see [2, Lemma 1.5.2] or [5, Lemma 3.14]. Hence also

$$(\omega, \xi) \mapsto \sup_n d_n((\xi, y), K(\omega)) = \rho(y, K(\omega, \xi))$$

is measurable so that $K(\omega, \xi)$ is indeed a random compact set. Finally, $h_{\omega, \xi}(K(\omega, \xi)) = K((\theta \times g)(\omega, \xi))$ follows from the observation

$$\begin{aligned} y \in K(\omega, \xi) &\Leftrightarrow (\xi, y) \in K(\omega) \\ &\Leftrightarrow T_\omega(\xi, y) = (g_\omega(\xi), h_{\omega, \xi}(y)) \in K(\theta\omega) \\ &\Leftrightarrow h_{\omega, \xi}(y) \in K((\theta \times g)(\omega, \xi)) = K(\theta\omega, g_\omega(\xi)). \end{aligned}$$

□

Proof of Lemma 3.8.

a) By Lemma 3.7, K is a random compact set over the base $(\Omega \times \Xi, \mathcal{F} \times \mathcal{B})$ so that there is a sequence $(a_k)_{k \in \mathbb{N}}$ of measurable maps $a_k : \Omega \times \Xi \rightarrow \mathbb{R}^d$ such that $K(\omega, \xi) = \text{closure}\{a_k(\omega, \xi) \mid k \in \mathbb{N}\}$ for all $(\omega, \xi) \in \Omega \times \Xi$ [5, Theorem III.30], see also [2, Proposition 1.6.3].

Let $\epsilon > 0$. For $n \in \mathbb{N}$ denote by \mathcal{L}_n the family of subsets of \mathbb{N} with n elements. Then the sets

$$V_n := \bigcup_{L \in \mathcal{L}_n} \bigcap_{k \in \mathbb{N}} \bigcup_{\ell \in L} \{(\omega, \xi) \in \Omega \times \Xi \mid \|a_k(\omega, \xi) - a_\ell(\omega, \xi)\| < \epsilon\}$$

are $\mathcal{F} \times \mathcal{B}$ -measurable, and $N_\epsilon(\omega, \xi) \leq n$ if and only if $(\omega, \xi) \in V_n$. This proves the measurability of N_ϵ .

b) Let $\omega \in \Omega$. In order to prove that the function $\xi \mapsto N_\epsilon(\omega, \xi)$ is upper semicontinuous, it suffices to observe that, for each $\alpha \in \mathbb{R}$, the set $\{\xi \in \Xi \mid N_\epsilon(\omega, \xi) < \alpha\}$ is open, because $K(\omega)$ is compact. □

4. RANDOM MINIMALITY AND RANDOM TRANSITIVITY

We start by collecting a few elementary, but useful facts.

Lemma 4.1. a) *If K is a random compact set, then $\text{int}(K)$ (the fibre-wise interior of K) is a random open set.*

b) *If $\theta \times g$ is a random homeomorphism and K is a random compact set, then $(\theta \times g)(K)$ is a random compact set with fibres $g_{\theta^{-1}\omega}(K(\theta^{-1}\omega))$.*

c) *If K_0, K_1, \dots are random compact sets, then $\bigcap_{n=0}^\infty K_n$ is a random compact set with fibres $\bigcap_{n=0}^\infty K_n(\omega)$.*

d) *If A is a random open or closed set, then $\pi_1(A) = \{\omega \in \Omega \mid A(\omega) \neq \emptyset\}$ is \mathcal{F} -measurable.*

Proof. a) See [6, Chapter 3].

b) There is a sequence a_0, a_1, a_2, \dots of measurable selections of K such that $K(\omega) = \text{closure}\{a_k(\omega) : k \in \mathbb{N}\}$ for all $\omega \in \Omega$ [5, Theorem III.30]. Then the maps $g_{\theta^{-1}}(a_k(\theta^{-1} \cdot))$ are measurable maps from Ω to Ξ and $g_{\theta^{-1}\omega}(K(\theta^{-1}\omega)) = \text{closure}\{g_{\theta^{-1}\omega}(a_k(\theta^{-1}\omega)) \mid k \in \mathbb{N}\}$, so that $(\theta \times g)(K)$ is a random compact set by [5, Theorem III.30] again.

c) This is part of [5, Proposition III.4].

d) For closed A , $\pi_1(A)$ is measurable by [2, Prop. 1.6.2]. If A is open, then $(\pi_1(A))^c = \{\omega \in \Omega \mid A^c(\omega) = \Xi\} = \bigcap_G \{\omega \in \Omega \mid A^c(\omega) \cap G \neq \emptyset\}$ where the intersection ranges over a countable base $\{G\}$ of the topology of Ξ , and this set is measurable by [2, Prop. 1.6.2] again. \square

Definition 4.2. Let $\theta \times g$ be a random homeomorphism on $\Omega \times \Xi$.

- a) A random point ξ is a measurable map $\xi : \Omega \rightarrow \Xi$.
- b) The random point ξ is a limit point of the sequence $(\xi_n)_{n \in \mathbb{N}}$ of random points, if for each $\omega \in \Omega$ the point $\xi(\omega) \in \Xi$ is a limit point of the sequence $(\xi_n(\omega))_{n \in \mathbb{N}}$.
- c) The forward orbit of the random point ξ is the sequence $(\xi_n)_{n \in \mathbb{N}}$ of random points defined by $\xi_n(\omega) = g_{\theta^{-n}\omega}^n(\xi(\theta^{-n}\omega))$. The backward orbit is defined analogously by replacing n with $-n$.
- d) The omega limit set of a random point ξ is the random compact set \mathcal{U}_ξ with fibres $\mathcal{U}_\xi(\omega) = \bigcap_{k \in \mathbb{N}} \text{closure}\{\xi_n(\omega) \mid n \geq k\}$ where $(\xi_n)_{n \in \mathbb{N}}$ is the orbit of ξ .

Remark 4.3. a) In view of [5, Theorem III.9] and Lemma 4.1, \mathcal{U}_ξ is indeed a random compact set.

b) \mathcal{U}_ξ is $(\theta \times g)$ -invariant.

c) A random point ζ is a limit point of the orbit of ξ if and only if ζ is a measurable selection of \mathcal{U}_ξ .

Recall the definition of a random minimal homeomorphism:

Definition 4.4. The random homeomorphism $\theta \times g$ on $\Omega \times \Xi$ is minimal, if each $(\theta \times g)$ -forward invariant random closed set K obeys the following dichotomy:

- either :** $K(\omega) = \Xi$ for \mathbb{P} -a.e. ω ,
- or :** $K(\omega) = \emptyset$ for \mathbb{P} -a.e. ω .

Lemma 4.5. Let $\theta \times g$ be a random homeomorphism on $\Omega \times \Xi$. The following are equivalent:

- (i) $\theta \times g$ is minimal.
- (ii) The dichotomy in Definition 4.4 holds for each $(\theta \times g)$ -backwards invariant random closed set K .
- (iii) The dichotomy in Definition 4.4 holds for each $(\theta \times g)$ -invariant random closed set K .
- (iv) For each random point ξ holds: $\mathcal{U}_\xi(\omega) = \Xi$ for \mathbb{P} -a.e. ω .

Proof. "(i) \Rightarrow (iii)" and "(ii) \Rightarrow (iii)" are trivial.

(iii) \Rightarrow (iv): For each random point ξ , \mathcal{U}_ξ is a non-empty $(\theta \times g)$ -invariant random compact set.

(iv) \Rightarrow (i): Let K be a $(\theta \times g)$ -forward invariant non-empty random closed set. It has at least one measurable selection ξ [5, Theorem III.9]. Denote its orbit by $(\xi_n)_{n \in \mathbb{N}}$. Then all ξ_n are measurable selections of K as well since $\xi_n(\omega) = g_{\theta^{-n}\omega}^n(\xi(\theta^{-n}\omega)) \in g_{\theta^{-n}\omega}^n(K(\theta^{-n}\omega)) \subseteq K(\omega)$. Hence, $\Xi = \mathcal{U}_\xi(\omega) \subseteq K(\omega)$ for \mathbb{P} -a.e. ω .

(iii) \Rightarrow (ii): Let K be a $(\theta \times g)$ -backward invariant random closed set. For each ω let $K'(\omega) = \bigcap_{n=0}^{\infty} (g_\omega^n)^{-1}(K(\theta^n\omega))$. As a decreasing intersection of non-empty random compact sets K' is a non-empty random compact set, see Lemma 4.1. It is invariant, because

$$g_\omega(K'(\omega)) = g_\omega \left(\bigcap_{n=1}^{\infty} (g_\omega^n)^{-1}(K(\theta^n\omega)) \right) = \bigcap_{n=0}^{\infty} (g_{\theta\omega}^n)^{-1}(K(\theta^{n+1}\omega)) = K'(\theta\omega).$$

Hence $K'(\omega) = \Xi$ for \mathbb{P} -a.e. ω . As $K'(\omega) \subseteq K(\omega)$, the same holds for $K(\omega)$. \square

It is tempting to characterise random transitive sets in the same way. We will see, however, that the situation is more complicated.

Definition 4.6. The random homeomorphism $\theta \ltimes g$ on $\Omega \times \Xi$ is transitive, if each $(\theta \ltimes g)$ -forward invariant random closed set K obeys the following dichotomy:

- either:** $K(\omega) = \Xi$ for \mathbb{P} -a.e. ω ,
or : $K(\omega)$ is nowhere dense for \mathbb{P} -a.e. ω .

In the same way as for Lemma 4.5, one proves

Lemma 4.7. Let $\theta \ltimes g$ be a random homeomorphism on $\Omega \times \Xi$. The following are equivalent:

- (i) $\theta \ltimes g$ is transitive.
- (ii) The dichotomy in Definition 4.6 holds for each $(\theta \ltimes g)$ -backwards invariant random closed set K .
- (iii) The dichotomy in Definition 4.6 holds for each $(\theta \ltimes g)$ -invariant random closed set K .

Recall that for a homeomorphisms g of a compact metric space Ξ , the following two statements are equivalent to topological transitivity: (1) Given any open sets $U, V \subseteq \Xi$ there exists $n \in \mathbb{N}$ with $g^n(U) \cap V \neq \emptyset$ and (2) there exists a point $x \in \Xi$ with dense orbit.

The first of these equivalent characterisations can easily be carried over to random dynamical systems. Recall that we say a random set R (open or compact) is *non-empty* if $\{\omega \in \Omega \mid R(\omega) \neq \emptyset\}$ has positive measure.

Lemma 4.8. A random homeomorphism $\theta \ltimes g$ on $\Omega \times \Xi$ is transitive if and only if

- (T1) for all non-empty random open sets $U, V \subseteq \Omega \times \Xi$ there exists $n \in \mathbb{N}$ such that $(\theta \ltimes g)^n(U) \cap V$ is non-empty.

Proof. Suppose $\theta \ltimes g$ is transitive and $U, V \subseteq \Xi$ are non-empty random open sets. Assume for a contradiction that

$$\mathbb{P}(\pi_1((\theta \ltimes g)^n(U) \cap V)) = 0$$

for all $n \in \mathbb{N}$. Let $\mathcal{U} = \bigcup_{n \in \mathbb{N}} (\theta \ltimes g)^n(U)$. Then \mathcal{U} is forward invariant and the set

$$\Omega' = \{\omega \in \Omega \mid \mathcal{U}(\omega) \cap V(\omega) = \emptyset\} = \bigcup_{n \in \mathbb{N}} \pi_1((\theta \ltimes g)^n(U) \cap V)$$

has measure zero. Hence, the backwards $(\theta \ltimes g)$ -invariant random compact set $K = (\Omega \times \Xi) \setminus \mathcal{U}$ contains $V' = V \setminus (\Omega' \times \Xi)$, but is disjoint from U . By Lemma 4.7(ii), this contradicts the transitivity of $\theta \ltimes g$.

Conversely, suppose that (T1) holds and K is a forward $(\theta \ltimes g)$ -invariant compact set such that $\text{int}(K(\omega)) \neq \emptyset$ \mathbb{P} -a.s. Let $V = (\Omega \times \Xi) \setminus K$ and assume for a contradiction that V is non-empty, such that we do not have $K(\omega) = \Xi$ for \mathbb{P} -a.e. $\omega \in \Omega$. Define the random open set U by $U(\omega) = \text{int}(K(\omega))$, see Lemma 4.1a. Then, by contradiction assumption, $U \subseteq K$ is non-empty, and due to the forward $(\theta \ltimes g)$ -invariance of K we have $(\theta \ltimes g)^n(U) \cap V \subseteq K \cap V = \emptyset$ for all $n \in \mathbb{N}$. This contradicts (T1). \square

In order to generalise (2) to the random situation, one might hope to characterise random transitivity by the existence of a random point ξ such that $\mathcal{U}_\xi(\omega) = \Xi$ for \mathbb{P} -a.e. ω . The following simple example shows that this cannot work: Let $\Omega = \Xi$ be a compact metric space and consider the random homeomorphism $\theta \ltimes g$ on $\Omega \times \Xi$ with a minimal homeomorphism $\theta : \Omega \rightarrow \Omega$, an ergodic probability \mathbb{P} on Ω with full topological support, and $g_\omega(\xi) = \xi$ for all (ω, ξ) . Except in the case of trivial Ξ , this is clearly not random transitive, because $(\Omega \times G)$ is an open $(\theta \ltimes g)$ -invariant set for each open $G \subset \Xi$. On the other hand, $\xi(\omega) = \omega$ defines a random point with $\xi_n(\omega) = \xi(\theta^{-n}\omega) = \theta^{-n}\omega$ for all ω and n . Then, by minimality of θ , we have $\mathcal{U}_\xi(\omega) = \Xi$ for all $\omega \in \Omega$.

Hence, in order to make sense of (2) in a random setting we need a refined concept of ‘dense orbit’. Given a random point ξ and a set $A \subseteq \Omega$ of positive measure, we call the restriction of ξ to A , denoted by $\xi^A : A \rightarrow \Xi$, $\omega \mapsto \xi(\theta)$, a *subsection* of ξ . We define $\xi_n^A : \theta^n(A) \rightarrow \Xi$ by $\xi_n^A(\omega) = g_{\theta^{-n}\omega}(\xi^A(\theta^{-n}\omega))$ and define the omega limit \mathcal{U}_{ξ^A} of ξ^A fibre-wise as $\mathcal{U}_{\xi^A}(\omega) = \text{closure}\{\xi_n^A(\omega) \mid n \in \mathbb{N} \text{ s.t. } \omega \in \theta^n(A)\}$.

Remark 4.9. The following observations are easy to prove.

- a) \mathcal{U}_{ξ^A} is a random compact set.

- b) \mathcal{U}_{ξ^A} is $(\theta \ltimes g)$ -invariant.
- c) Item (iv) in Lemma 4.5 can be replaced by
 - (iv)' $\mathcal{U}_{\xi^A}(\omega) = \Xi$ \mathbb{P} -a.s. for every subsection ξ^A of any random point ξ .

Using this concept, we have the following.

Lemma 4.10. *Suppose for the random homeomorphism $\theta \ltimes g$ there exists a random point ξ such that $\mathcal{U}_{\xi^A}(\omega) = \Xi$ \mathbb{P} -a.s. for all subsections ξ^A of ξ . Then $\theta \ltimes g$ is transitive.*

Proof. Let ξ be the random point with the above property. Suppose K is a $(\theta \ltimes g)$ -invariant random compact set such that $\{\omega \in \Omega \mid \text{int}(K(\omega)) \neq \emptyset\}$ has positive measure. Let $U \subseteq K$ be the non-empty random open set defined by $U(\omega) = \text{int}(K(\omega))$.

As $\mathcal{U}_{\xi}(\omega) = \Xi$ for \mathbb{P} -a.e. $\omega \in \Omega$, the set

$$\{\omega \in \Omega \mid \exists n \in \mathbb{N} : \xi_n(\omega) \in U(\omega)\} = \bigcup_{n \in \mathbb{N}} \{\omega \in \Omega \mid \xi_n(\omega) \in U(\omega)\}$$

has full measure. Hence, there exists $n \in \mathbb{N}$ such that $A' = \{\omega \in \Omega \mid \xi_n(\omega) \in U(\omega)\}$ has positive measure. Then $A = \theta^{-n}(A')$ satisfies $\xi^A(\omega) \in K(\omega)$ for all $\omega \in A$, and consequently $\mathcal{U}_{\xi^A} \subseteq K$. By assumption, this implies $K(\omega) = \Xi$ \mathbb{P} -a.s. Since K was an arbitrary $(\theta \ltimes g)$ -invariant random compact set, Lemma 4.7(iii) implies that $\theta \ltimes g$ is transitive. \square

This suggests that a natural way to generalise (2) would be to show that random transitivity is equivalent to the existence of a random point ξ whose subsections all have dense orbit (in the sense that $\mathcal{U}_{\xi^A} = \Xi$ for \mathbb{P} -a.e. $\omega \in \Omega$). However, we are at the moment not able to prove this, nor to give a counterexample. A positive result may depend on additional assumptions on the base $(\Omega, \mathcal{B}, \mathbb{P}, \theta)$. We thus have to leave the following problem open.

Problem 4.11. Show that under suitable assumptions on $(\Omega, \mathcal{B}, \mathbb{P}, \theta)$ a transitive random homeomorphism $\theta \ltimes g$ has a random point ξ whose subsections all have dense orbit.

Finally, we want to mention that a natural class of examples of minimal random homeomorphisms is given by random circle rotations, although it needs a little work to give a rigorous proof of their minimality. We close this section by providing a sketch of the argument. Given a measure-preserving dynamical system $(\Omega, \mathcal{B}, \mathbb{P}, \theta)$ and a measurable function $\alpha : \Omega \rightarrow \mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, we let $\Xi = \mathbb{S}^1$ and define a random homeomorphism $\theta \ltimes g^\tau$ with parameter $\tau \in \mathbb{S}^1$ by

$$(4.1) \quad g_\omega^\tau(z) = \alpha(\omega)\tau z.$$

It is easy to see that $\mathbb{P} \times \text{Leb}_{\mathbb{S}^1}$ belongs to $\mathcal{M}_{\mathbb{P}}(\theta \ltimes g^\tau)$. Further, using the random Krylov-Bogolyubov Procedure [2, Theorem 1.5.8] it is possible to show that if $\mathbb{P} \times \text{Leb}_{\mathbb{S}^1}$ is the unique measure in $\mathcal{M}_{\mathbb{P}}(\theta \ltimes g^\tau)$, then $\theta \ltimes g$ is minimal. An old result of Furstenberg [7, Lemma 2.1] states that the existence of further measures in $\mathcal{M}_{\mathbb{P}}(\theta \ltimes g^\tau)$ implies that the Koopman operator of θ has a unimodular measurable generalised eigenfunction R in the sense that

$$R(\theta\omega) = (\tau\alpha(\omega))^n R(\omega) \quad \text{for some } n \in \mathbb{Z}.$$

Note that while the result in [7] is stated for continuous and uniquely ergodic base transformations θ , the proof also works in the general case. If, for a given $n \in \mathbb{Z}$, there are two numbers τ and $\tilde{\tau}$ for which this identity holds (with eigenfunctions R and \tilde{R}), then $\tilde{R}R^{-1}$ is an eigenfunction of the Koopman operator with eigenvalue $(\tilde{\tau}\tau^{-1})^n$. Now, if θ is weakly mixing, then the only L^2 -eigenvalue of the associated Koopman operator is 1, so $\tilde{\tau}\tau^{-1}$ is a root of unity. As a consequence, there exist only countably many $\tau \in \mathbb{S}^1$ such that $\mathcal{M}_{\mathbb{P}}(\theta \ltimes g)$ contains more than one measure. Hence, g^τ is minimal for all but countably many $\tau \in \mathbb{S}^1$.

Basically the same argument works when θ is a minimal rotation of a d -dimensional torus. In this case, it follows easily from Fourier analysis that any iterate of θ can only have countably many L^2 -eigenvalues, such that again g^τ is minimal for all but countably many $\tau \in \mathbb{S}^1$.

REFERENCES

- [1] V. Anagnostopoulou, T. Jäger, and G. Keller. A model for the non-autonomous hopf bifurcation. In preparation.

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- [2] L. Arnold. *Random Dynamical Systems*. Springer, 1998.
 - [3] J.-P. Aubin and H. Frankowska. *Set-Valued Analysis*. Birkhäuser, 1990.
 - [4] Y. Cao, S. Luzzatto, and I. Rios. Uniform hyperbolicity for random maps with positive Lyapunov exponents. *Proceedings of the American Mathematical Society*, 136(10):3591–3600, May 2008.
 - [5] C. Castaing and M. Valadier. *Convex Analysis and Measurable Multifunctions*, volume 580 of *Lecture Notes in Mathematics*. Springer, 1977.
 - [6] H. Crauel. *Random Probability Measures on Polish Spaces*, volume 11 of *Stochastics Monographs*. Taylor & Francis, 2002.
 - [7] H. Furstenberg. Strict ergodicity and transformation of the torus. *Am. J. Math.*, 83:573–601, 1961.
 - [8] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1997.
 - [9] G. Keller. *Equilibrium states in ergodic theory*. Cambridge Univ Pr, 1998.
 - [10] H. Keller and G. Ochs. Numerical approximation of random attractors. In *Stochastic Dynamics*, pages 93–115. Springer, 1999.
 - [11] J. Stark and R. Sturman. Semi-uniform ergodic theorems and applications to forced systems. *Nonlinearity*, 13(1):113–143, 2000.